1. Unit: Logic and Symbolic Reasoning

Exercise 1.1 (T_P-Operator and Resolution)

Consider the following program (cf. slides from the lecture):

$$\begin{array}{ll} P = \{ & \mathsf{border}(a,d).\ \mathsf{border}(a,h).\ \mathsf{border}(a,i).\ \mathsf{border}(d,f).\ \mathsf{border}(i,f). \\ & \mathsf{border}(ch,f).\ \mathsf{border}(ch,a).\ \mathsf{border}(ch,d).\ \mathsf{border}(ch,i).\ \mathsf{border}(e,f).\ \mathsf{border}(p,e). \\ & \mathsf{border}(h,ua).\ \mathsf{border}(ua,r).\ \mathsf{border}(ra,br).\ \mathsf{border}(bol,ra).\ \mathsf{border}(bol,br). \\ & \mathsf{border}(Y,X) \leftarrow \mathsf{border}(X,Y). \\ & \mathsf{reachable}(X,Y) \leftarrow \mathsf{border}(X,Y). \\ & \mathsf{reachable}(X,Y) \leftarrow \mathsf{reachable}(X,Z),\ \mathsf{border}(Z,Y). \quad \} \end{array}$$

- Give $T_P^0(\emptyset)$, $T_P^1(\emptyset)$, $T_P^2(\emptyset)$, ..., $T_P^\omega(\emptyset)$.
- for any derived fact $\mathsf{reachable}(c_1, c_2) \in T_P^\omega(\emptyset)$, characterize the least i such that $\mathsf{reachable}(c_1, c_2) \in T_P^i(\emptyset)$.
- illustrate the effect of stratification by adding the rule $\mathsf{unreachable}(X,Y) \leftarrow \mathsf{country}(X), \mathsf{country}(Y), \neg\mathsf{reachable}(X,Y)$.
- prove $\mathsf{reachable}(e,h)$ by resolution.

Take only the following subset of the facts:

$$\begin{array}{ll} \{ & \mathsf{border}(a,h).\ \mathsf{border}(a,i).\ \mathsf{border}(i,f).\ \mathsf{border}(ch,f). \\ & \mathsf{border}(ch,a).\ \mathsf{border}(ch,i).\ \mathsf{border}(e,f).\ \mathsf{border}(p,e).\ \} \end{array}$$

${T_P}^0(\emptyset) = \emptyset$.
${T_P}^1(\emptyset) = T_P(\emptyset)$: all facts (as listed above).
${T_P}^2(\emptyset) = T_P({T_P}^1(\emptyset))$: facts + all applications of symmetry rule for borders + base case for reachable: ${T_P}^2(\emptyset) = {T_P}^1(\emptyset) \cup$

$$\begin{array}{ll} \{ & \mathsf{border}(h,a),\ \ \mathsf{border}(i,a),\ \ \mathsf{border}(f,i),\ \ \mathsf{border}(f,ch), \\ & \mathsf{border}(a,ch),\ \ \mathsf{border}(i,ch),\ \ \mathsf{border}(f,e),\ \ \mathsf{border}(e,p)\ \} \cup \\ \{ & \mathsf{reachable}(a,h),\ \ \mathsf{reachable}(a,i),\ \ \mathsf{reachable}(i,f),\ \ \mathsf{reachable}(ch,f), \\ & \mathsf{reachable}(ch,a),\ \ \mathsf{reachable}(ch,i),\ \ \mathsf{reachable}(e,f),\ \ \mathsf{reachable}(p,e)\ \} \end{array}$$

border is now symmetric. reachable contains the non-symmetric neighboring pairs that have been given by the original facts.

${T_P}^3(\emptyset)$: the base case for reachable now completes the neighboring countries (indicated by $_1$). The recursive rule is applied applied to the available results from the previous step (indicated by $_2$, adding all neighbors of countries reachable there, including the country itself).

${T_P}^4(\emptyset)$ etc: the neighbors of the reachable countries from the previous round are added.

Table for reachable:

	P	E	F	CH	A	I	H
${T_P}^1$							
${T_P}^2$	E	F		F,A,I	H,I	F	
${T_P}^3$	E,P_2,F_2	F,P_1,E_2,I_2,CH_2	I_1,CH_1,E_1	F,A,I,CH_2,E_2,H_2	H,I,CH_1,A_2,F_2	F,A_1,CH_1,I_2,E_2	A_1
${T_P}^4$	+I	+A	+F,+A,+P		+E	+P	+H,+CH,+I
${T_P}^5$	+A	+H	+H		+P		+F
${T_P}^6$	+H						+E
${T_P}^7$							+P
${T_P}^8$							

With ${T_P}^8(\emptyset) = {T_P}^8(\emptyset) =: {T_P}^\omega(\emptyset)$, the process ends.

Characterization:

- symmetric borders: in T_P^3 completed.
- let $i - neighbor(x,y)$ denote that y is reachable from x by crossing at least i borders.
  Then, the i-neighbors are completed in step T_P^{i+2}. More exactly: if the “first” border to cross is already given in the right direction in the facts, these i-neighbors are already contained in step T_P^{i+1}.
  Thus, $i+2$ steps are needed.

Optimization: The rule $\mathsf{reachable}(X,Y) \leftarrow \mathsf{reachable}(X,Z), \underline{\mathsf{reachable}}(Z,Y)$.
would reduce the overall number of steps to $log_2(i)+2$.

Stratification: Having the rule in the same set would fire it in T_P^1, adding unreachable(X,Y) for all pairs. Firing it only after the first stratum is completed, i.e., when ${T_P}^{\omega}(\emptyset)$ is computed adds in this case nothing (but would e.g. add unreachable(D,USA) in the complete database).

Resolution: (use (e,f), $(i,f)^{-1}$, $(a,i)^{-1}$, (a,h))

Clauses: (1) – negated claim: $\{\neg r(e,h)\}$
(2) rule: $\{r(X,Y), \neg r(X,Z), \neg b(Z,Y)\}$
(3) rule: $\{r(X,Y), \neg b(X,Y)\}$
(4) rule: $\{b(X,Y), \neg b(Y,X)\}$
facts to be used: (5) $\{b(e,f)\}$ (6) $\{b(i,f)\}$ (7) $\{b(a,i)\}$ (8) $\{b(a,h)\}$

(1) with (2) $[X \mapsto e, Y \mapsto h]$:
(9) $\{\neg r(e,Z), \neg b(Z,h)\}$ (means: for all Z, at least one of these holds)
(9) with (8) $[Z \mapsto a]$: (10) $\{\neg r(e,a)\}$
(10) with (2) $[X \mapsto e, Y \mapsto a]$:
(11) $\{\neg r(e,Z), \neg b(Z,a)\}$
(11) with (4) $[X \mapsto Z, Y \mapsto a]$:
(12) $\{\neg r(e,Z), \neg b(a,Z)\}$
(12) with (7) $[Z \mapsto i]$: (13) $\{\neg r(e,i)\}$
(13) with (2) $[X \mapsto e, Y \mapsto i]$:
(14) $\{\neg r(e,Z), \neg b(Z,i)\}$
(14) with (4) $[X \mapsto Z, Y \mapsto i]$:
(15) $\{\neg r(e,Z), \neg b(i,Z)\}$
(15) with (6) $[Z \mapsto f]$: (16) $\{\neg r(e,f)\}$
(16) with (3) $[X \mapsto e, Y \mapsto f]$: (17) $\{\neg b(e,f)\}$
(17) with (5) $\Box$.

I knew what I was doing ... a real prover will run into lots of wrong choices, backtracking etc. Important strategy: (i) ground resolution, (ii) unit resolution: if one resolvent is a unary clause, there is no growth.
Note that also a resolution of (2) with itself (renamed)

$$\begin{array}{ll} (2a) & \{r(X_1,Y_1), \overline{\neg r(X_1,Z_1)}, \neg b(Z_1,Y_1)\} \\ (2b) & \{\underline{r(X_1,Z_1)}, \neg r(X_1,Z_2), \neg b(Z_2,Z_1)\} \\ \hline & \{r(X_1,Y_1), \neg r(X_1,Z_2), \neg b(Z_2,Z_1), \neg b(Z_1,Y_1)\} \end{array}$$

would be possible. Resolving/expanding this one more time results in a clause that contains 3 intermediate countries which could be resolved against the (symmetric) borders e-f-i-a-h.